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Exactly solvable quantum mechanical models with infinite renormalization of the wavefunction

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Abstract
The main difficulty of quantum field theory is the problem of divergences and renormalization. However, realistic models of quantum field theory are renormalized within the perturbative framework only. It is important to investigate renormalization beyond perturbation theory. However, known models of constructive field theory do not contain such difficulties as infinite renormalization of the wavefunction. In this paper an exactly solvable quantum mechanical model with such a difficulty is constructed. This model is a simplified analogue of the large-N approximation to the $\Phi\varphi^a\varphi^a$-model in six-dimensional spacetime. It is necessary to introduce an indefinite inner product to renormalize the theory. The mathematical results of the theory of Pontriagin spaces are essentially used. It is remarkable that not only the field but also the canonically conjugated momentum become well defined operators after adding counterterms.

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1. Introduction

An essential feature of realistic models of QFT (such as quantum electrodynamics, Yang–Mills theory etc) is the property of infinite renormalization of the wavefunction. This difficulty leads to problems of canonical quantization of the theory. Since the coefficient z of the term $\partial_\mu\varphi\partial_\mu\varphi$ of the Lagrangian diverges, the momentum canonically conjugated to the field φ should be related to the time derivative of the field φ as

$$\pi = z\dot{\varphi}.$$

If we believe φ to be an operator-valued distribution [1], its derivative can be also interpreted in the same way. Therefore, the momentum cannot be viewed even as an operator distribution because of the infinite coefficient z.

Infinite renormalization of the wavefunction is a serious difficulty in the constructive field theory [2, 3]. Rigorous construction of mathematical models of QFT has been successful for models with finite z only. The $z = \infty$ case leads to serious difficulties (see, e.g., [4]).

This paper deals with the exactly solvable quantum mechanical model with infinite renormalization of the wavefunction. The Lagrangian of the model is formally written as

$$L = \frac{z\dot{Q}^2}{2} - \frac{m^2Q^2}{2} + \sum_{k=1}^{\infty}\left(\frac{\dot{q}_k^2}{2} - \frac{\Omega_k^2 q_k^2}{2}\right) - gQ\sum_{k=1}^{\infty}\mu_k q_k. \tag{1}$$

Here μ_k are real quantities, while Ω_k, $k = \overline{1,\infty}$, is an increasing sequence of real positive numbers.

Renormalization properties of the model (1) depend on the large-k behaviour of the sequence μ_k.

(a) If $\sum_k \mu_k^2/\Omega_k < \infty$, the model is quantized in a standard way: one constructs the Hamiltonian, introduces the creation and annihilation operators

$$q_k = \frac{a_k^+ + a_k^-}{\sqrt{2\Omega_k}} \qquad \dot{q}_k = \mathrm{i}\sqrt{\frac{\Omega_k}{2}}(a_k^+ - a_k^-)$$

with the commutation relations

$$[a_k^\pm, a_m^\pm] = 0 \qquad [a_k^-, a_m^+] = \delta_{km}$$

and shows the obtained Hamiltonian to be a correctly defined self-adjoint operator.

(b) If $\sum_k \mu_k^2/\Omega_k = \infty$ but $\sum_k \mu_k^2/\Omega_k^2 < \infty$, renormalization of vacuum energy is necessary. The Hamiltonian is a self-adjoint operator with nontrivial domain.

(c) If $\sum_k \mu_k^2/\Omega_k^2 = \infty$ but $\sum_k \mu_k^2/\Omega_k^3 < \infty$, it is necessary to perform renormalization of m^2. The vacuum divergences arise. They can be removed by the Faddeev transformation [5].

(d) If $\sum_k \mu_k^2/\Omega_k^3 = \infty$ but $\sum_k \mu_k^2/\Omega_k^4 < \infty$, there is an additional difficulty: the Stueckelberg divergences [6] arise. They can be removed by the Faddeev-type transformation [7].

(e) If $\sum_k \mu_k^2/\Omega_k^4 = \infty$ but $\sum_k \mu_k^2/\Omega_k^6 < \infty$, one should perform infinite renormalization of the wavefunction z.

(f) If $\sum_k \mu_k^2/\Omega_k^6 = \infty$, it is necessary to add new counterterms to the Lagrangian.

In this paper the model (1) is mathematically constructed for the most interesting case (e). The cases (a)–(d) are more trivial and can be investigated according to [8–10].

It is interesting that the $z = \infty$ case leads to an indefinite inner product in the state space analogously to the Lee model [11–15], the perturbative Hamiltonian QFT [16] and the strongly singular potentials in quantum mechanics [17–20]. The state space is the Fock space associated with the one-particle Pontriagin space. The results of the general mathematical theory of Pontriagin spaces [21–24] are essentially used.

It will be shown that the expressions

$$Q(t) \qquad zQ(t) - \sum_{k=1}^{\infty}\frac{g\mu_k}{\Omega_k^2}q_k(t) \tag{2}$$

may be both viewed as operator distributions. Differentiating the second expression, we obtain that the momentum $P(t)$ canonically conjugated to $Q(t)$ becomes an operator distribution after adding a counterterm:

$$P(t) - \sum_{k=1}^{\infty}\frac{g\mu_k}{\Omega_k^2}p_k(t).$$

The model of the type (1) arises in the quantum probability theory [25, 26] and in the condensed-matter theory ('polaron model' [27]).

It is also an analogue of the model $\Phi\varphi^a\varphi^a$ of a large number of fields which is viewed in the leading order of $1/N$-expansion. Namely, consider the theory of N fields φ^a interacting with the field Φ in the $(d+1)$-dimensional spacetime. The Lagrangian of the theory is

$$\mathcal{L} = \sum_{a=1}^{N} : \left(\frac{1}{2}\partial_\mu\varphi^a\partial_\mu\varphi^a - \frac{\mu^2}{2}\varphi^a\varphi^a\right) : + \frac{z}{2}\partial_\mu\Phi\partial_\mu\Phi - \frac{M^2}{2}\Phi^2 - \frac{g}{\sqrt{N}} : \left(\sum_{a=1}^{N}\varphi^a\varphi^a\right) : \Phi.$$

Analogously to [28] (see also [29–32]), introduce the 'collective fields', being the operators of creation and annihilation of pairs of particles

$$A^{\pm}_{kp} = \frac{1}{\sqrt{2N}}\sum_{a=1}^{N} b^{\pm a}_{k} b^{\pm a}_{p}$$

where $b^{\pm a}_{k}$ is a creation–annihilation operator of the particle with momentum $\boldsymbol{k}$, which corresponds to the field φ^a.

We will consider the states of the N-field theory which depend on the large parameter N as follows:

$$\sum_n \int \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{p}_1\cdots\mathrm{d}\boldsymbol{k}_n\,\mathrm{d}\boldsymbol{k}_n\,A^{+}_{k_1p_1}\cdots A^{+}_{k_np_n}\chi^{n}_{k_1p_1\ldots k_np_n}\Psi \tag{3}$$

with regular as $N\to\infty$ coefficient functions χ^n and such a vector Ψ that it does not contain the particles corresponding to the fields φ^a.

Note that operators of the form

$$\int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{p}\frac{1}{\sqrt{N}}\sum_{a=1}^{N} b^{+a}_{k} b^{-a}_{p}\varphi_{kp}$$

multiply the norm of the state (3) by the quantity $\mathrm{O}(N^{-1/2})$. Therefore, they can be neglected as $N\to\infty$. In this approximation

$$[A^{-}_{k_1p_1}; A^{+}_{k_2p_2}] \simeq \tfrac{1}{2}(\delta_{k_1k_2}\delta_{p_1p_2} + \delta_{k_1p_2}\delta_{k_2p_1}).$$

Consider the free Hamiltonian $H_0 = \int \mathrm{d}\boldsymbol{k}\,\omega_k\sum_{a=1}^{N} b^{+a}_{k} b^{-a}_{k}$, where $\omega_k = \sqrt{\boldsymbol{k}^2+\mu^2}$. If we consider the states of the form (3) only, it coincides with the operator

$$\int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{p}\,A^{+}_{kp}(\omega_k+\omega_p)A^{-}_{kp}.$$

The operator $\frac{1}{\sqrt{N}}\sum_{a=1}^{N}\varphi^a(\boldsymbol{x})\varphi^a(\boldsymbol{x})$ is approximately equal to

$$\frac{\sqrt{2}}{(2\pi)^d}\int\frac{\mathrm{d}\boldsymbol{k}}{\sqrt{2\omega_k}}\frac{\mathrm{d}\boldsymbol{p}}{\sqrt{2\omega_p}}(A^{+}_{kp}\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{p})\boldsymbol{x}} + A^{-}_{kp}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{p})\boldsymbol{x}}).$$

The leading order for the Hamiltonian in $1/N$ is

$$\begin{aligned} H = {}& \int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{p}\,A^{+}_{kp}(\omega_k+\omega_p)A^{-}_{kp} + \int \mathrm{d}\boldsymbol{x}\left(\frac{1}{2z}\Pi^2(\boldsymbol{x}) + \frac{z}{2}(\nabla\Phi)^2(\boldsymbol{x}) + \frac{M^2}{2}\Phi^2\right) \\ & + \frac{\sqrt{2}g}{(2\pi)^d}\int \mathrm{d}\boldsymbol{x}\left[\int\frac{\mathrm{d}\boldsymbol{k}}{\sqrt{2\omega_k}}\frac{\mathrm{d}\boldsymbol{p}}{\sqrt{2\omega_p}}(A^{+}_{kp}\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{p})\boldsymbol{x}} + A^{-}_{kp}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{p})\boldsymbol{x}})\right]\Phi(\boldsymbol{x}). \end{aligned} \tag{4}$$

Equation (4) can be also obtained from the third-quantized approach [10, 33, 34].

We see that generalization of the model (1) with the Hamiltonian

$$H = \sum_i\left(\frac{P_i^2}{2Z} + \frac{Z}{2}M_i^2Q_i^2\right) + \sum_k\left(\frac{p_k^2}{2} + \frac{\Omega_k^2q_k^2}{2}\right) + g\sum_i Q_i\sum_k\mu^{(i)}_k q_k$$

resembles Hamiltonian (4) if one considers $(\boldsymbol{k}, \boldsymbol{p})$ instead of k, $\boldsymbol{x}$ instead of i and integrals instead of sums. The field operator $\Phi(\boldsymbol{x}, t)$ and Heisenberg creation–annihilation operators related to the composed field $\varphi^a\varphi^a$ are analogues of $Q_i(t)$ and $q_k(t)$ correspondingly.

This paper is organized as follows. Section 2 deals with diagonalization of the Hamiltonian. In section 3 renormalization of the model is performed. Section 4 deals with constructing operators $Q(t)$ and $q_k(t)$, which are analogues of field operators of the model (4), and justifying the hypothesis that expression (2) corresponds to correctly defined operator distributions in the renormalized theory.

2. Investigation of the regularized model

Let us quantize the model (1). Let Λ be a positive integer regularization parameter. Perform a substitution $\mu_k \to \mu_k^\Lambda$, where $\mu_k^\Lambda = \mu_k$ at $k < \Lambda$ and $\mu_k^\Lambda = 0$ at $k > \Lambda$. Let z and m^2 be also Λ dependent, z_Λ and m^2_Λ. Then the Hamiltonian takes the form

$$H = \sum_{mn} \left(\tfrac{1}{2} p_m Z^{-1}_{\Lambda,mn} p_n + \tfrac{1}{2} q_m M^2_{\Lambda,mn} q_n\right) \tag{5}$$

where $Z_{\Lambda,mn}$, $m, n = \overline{0, \infty}$ and $M^2_{\Lambda,mn}$ are matrices of the form

$$Z_\Lambda = \begin{pmatrix} z_\Lambda & 0 \\ 0 & 1 \end{pmatrix} \qquad M^2_\Lambda = \begin{pmatrix} m^2_\Lambda & g\mu^\Lambda \\ g\mu^\Lambda & \Omega^2 \end{pmatrix}$$

$q_0 \equiv Q$, $p_0 \equiv P$ is a momentum conjugated to Q and p_k are momenta conjugated to q_k.

Suppose that there exist operators $(Z^{-1}_\Lambda M^2_\Lambda)^{\pm 1/4}$. After transformation

$$\begin{aligned} q_m &= \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} ((Z^{-1}_\Lambda M^2_\Lambda)^{-1/4} Z^{-1}_\Lambda)_{mn} (b^+_n + b^-_n) \\ p_m &= \frac{\mathrm{i}}{\sqrt{2}} \sum_{n=0}^{\infty} (M^2_\Lambda Z^{-1}_\Lambda)^{1/4}_{mn} (b^+_n - b^-_n) \end{aligned} \tag{6}$$

the Hamiltonian (5) takes the form

$$H_\Lambda = \sum_{mn=0}^{\infty} b^+_m [(Z^{-1}_\Lambda M^2_\Lambda)^{1/2} Z^{-1}_\Lambda]_{mn} b^-_n \tag{7}$$

up to an additive constant interpreted as vacuum energy, which can be removed by renormalization.

The canonical commutation relations are written as

$$[b^\pm_m, b^\pm_n] = 0 \qquad [b^-_m, b^+_n] = Z_{\Lambda,mn}. \tag{8}$$

Choose the Fock representation for the operators $b^\pm_m$. Any state vector can be presented as

$$\Psi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{k_1 \ldots k_n} \psi^{(n)}_{k_1 \ldots k_n} b^+_{k_1} \cdots b^+_{k_n} |0\rangle \tag{9}$$

where $|0\rangle$ is a vacuum state, $b^-_k|0\rangle = 0$ and $\psi^{(n)}_{k_1\ldots k_n}$ are functions of $k_1 \ldots k_n$ which are symmetric with respect to their transpositions. Relations (8) imply that the inner product can be presented as

$$(\Phi, \Phi) = \sum_{n=0}^{\infty} \sum_{k_1 \ldots k_n p_1 \ldots p_n} \psi^{(n)*}_{p_1 \ldots p_n} Z_{\Lambda, p_1 k_1} \cdots Z_{\Lambda, p_n k_n} \psi^{(n)}_{k_1 \ldots k_n} \tag{10}$$

while the Hamiltonian operator acts as

$$(H_\Lambda\psi)^{(n)}_{k_1\ldots k_n} = \sum_{i=1}^{n}\sum_{p_i}(Z_\Lambda^{-1}M_\Lambda^2)^{1/2}_{k_i p_i}\psi^{(n)}_{k_1\ldots k_{i-1}p_i k_{i+1}\ldots k_n}. \tag{11}$$

The evolution operator can be written as

$$(\mathrm{e}^{-\mathrm{i}H_\Lambda t}\psi)^{(n)}_{k_1\ldots k_n} = \sum_{i=1}^{n}\sum_{p_1\ldots p_n}(\mathrm{e}^{-\mathrm{i}(Z_\lambda^{-1}M_\Lambda^2)^{1/2}t})_{k_1 p_1}\cdots(\mathrm{e}^{-\mathrm{i}(Z_\Lambda^{-1}M_\Lambda^2)^{1/2}t})_{k_n p_n}\psi^{(n)}_{p_1\ldots p_n}.$$

By $\mathcal{P}_\Lambda$ we denote the space of sets ψ_k, $k=\overline{0,\infty}$, with the indefinite inner product

$$\langle\psi,\psi\rangle_\Lambda = (\psi, Z_\Lambda\psi). \tag{12}$$

We see that the state space is the Fock space associated with $\mathcal{P}_\Lambda$:

$$\mathcal{F}(\mathcal{P}_\Lambda) = \oplus_{n=0}^{\infty}\mathcal{P}_\Lambda^{\vee n}$$

where $\mathcal{P}_\Lambda^{\vee n}$ is the nth symmetric tensor degree of the space $\mathcal{P}_\Lambda$ [1]. The evolution operator is

$$\mathrm{e}^{-\mathrm{i}H_\Lambda t} = \oplus_{n=0}^{\infty}(\exp(-\mathrm{i}(Z_\Lambda^{-1}M_\Lambda^2)^{1/2}t))^{\otimes n}.$$

3. Problem of renormalization

(1) There are several ways to renormalize a quantum field theory model. For example, one can first evaluate such vacuum expectations as Green or Wightman functions [1, 35], r-functions [36] or S-matrix coefficient functions [37,38] for the regularized theory and consider the limit $\Lambda\to\infty$ for these quantities. Then the Wightman reconstruction theorem [35] or its analogue can be applied.

In the approach based on the dynamical Hamiltonian equations of motion rather than the S-matrix another way to perform a limit $\Lambda\to\infty$ can be used. If H_Λ is a regularized Hamiltonian acting in the Hilbert space $\mathcal{H}$, one can try to choose a unitary operator $T_\Lambda:\mathcal{H}\to\mathcal{H}$ ('dressing transformation' [2, 5]) singular as $\Lambda\to\infty$ such that the operator

$$T_\Lambda^{+}\mathrm{e}^{-\mathrm{i}H_\Lambda t}T_\Lambda$$

has a strong limit as $\Lambda\to\infty$. The limit

$$U(t) = s-\lim_{\Lambda\to\infty}T_\Lambda^{+}\mathrm{e}^{-\mathrm{i}H_\Lambda t}T_\Lambda \tag{13}$$

can be interpreted as an evolution operator in the renormalized theory. The difficulty of our case is that different spaces $\mathcal{F}(\mathcal{P}_\Lambda)$ are considered at different values of Λ. Another essential feature is that $\mathcal{F}(\mathcal{P}_\Lambda)$ are not Hilbert spaces but indefinite inner product spaces. Therefore, the requirement (13) should be modified.

We say that renormalization is performed if:

(i) a Hilbert inner product is introduced on $\mathcal{F}(\mathcal{P}_\Lambda)$;
(ii) a Pontriagin space $\mathcal{L}$ ('renormalized state space') is specified;
(iii) an operator $T_\Lambda:\mathcal{L}\to\mathcal{F}(\mathcal{P}_\Lambda)$ is defined;
(iv) for some operator $U(t):\mathcal{L}\to\mathcal{L}$ ('renormalized evolution operator') and any vector $\Psi=(\psi_0,\psi_1,\ldots,\psi_n,0,0,\ldots)$

$$||T_\Lambda U(t)\Psi - \mathrm{e}^{-\mathrm{i}H_\Lambda t}T_\Lambda\Psi|| \to_{\Lambda\to\infty} 0. \tag{14}$$

Condition (14) is a modification of condition (13). Its physical meaning is the following. Suppose that $T_\Lambda \Psi$ is chosen to be an initial state in the regularized theory. Then the state at time t can be approximated by the vector $T_\Lambda U(t)\Psi$. The operator $U(t)$ can be viewed as a renormalized evolution operator.

Note also that relation (14) means that the operator $\mathrm{e}^{-\mathrm{i}H_\Lambda t} : \mathcal{F}(\mathcal{P}_\Lambda) \to \mathcal{F}(\mathcal{P}_\Lambda)$ tends to $U(t) : \mathcal{L} \to \mathcal{L}$ in a generalized strong sense [39, 40].

We will choose $\mathcal{L} = \mathcal{F}(\mathcal{P})$,

$$T_\Lambda = \oplus_{n=0}^{\infty} (P_\Lambda)^{\otimes n}$$

for some Pontriagin space $\mathcal{P}$ and some operator $P_\Lambda : \mathcal{P} \to \mathcal{P}_\Lambda$.

(2) To introduce a Hilbert inner product on $\mathcal{F}(\mathcal{P}_\Lambda)$, it is sufficient to introduce it on $\mathcal{P}_\Lambda$. The standard method is the following [22]. Let e_Λ be an element of $\mathcal{P}_\Lambda$ such that $\langle e_\Lambda, e_\Lambda\rangle_\Lambda < 0$. Denote by $[e_\Lambda]$ the one-dimensional space $\{\lambda e_\Lambda | \lambda \in \boldsymbol{C}\}$, while $[e_\Lambda]^\perp$ is the space of all vectors ψ such that $\langle \psi, e_\Lambda\rangle_\Lambda = 0$. If the inner product is positively definite on $[e_\Lambda]^\perp$, the indefinite inner product space is of the type Π_1 [21, 22]. We see that this is true for the case $e_\Lambda = (1, 0, \ldots)$, provided that $z_\Lambda < 0$ (this condition will be shown to be satisfied at sufficiently large Λ). The positive definiteness of the inner product on $[e_\Lambda]^\perp$ for arbitrary e_Λ is a corollary of the general theory of Pontriagin spaces [22].

The Hilbert inner product is introduced as

$$(f, g)_{e_\Lambda} = \langle f, g\rangle_\Lambda + 2\frac{\langle f, e_\Lambda\rangle_\Lambda \langle e_\Lambda, g\rangle_\Lambda}{|\langle e_\Lambda, e_\Lambda\rangle_\Lambda|}. \tag{15}$$

One can notice that $(f, g)_{e_\Lambda} = \langle f, g\rangle_\Lambda$ if $f, g \perp e_\Lambda$, $(f, g)_{e_\Lambda} = -\langle f, g\rangle_\Lambda$ if $f, g \in [e_\Lambda]$ and $(f, g)_{e_\Lambda} = 0$ if $f \in [e_\Lambda]$, $g \perp e_\Lambda$. All topologies on $\mathcal{P}_\Lambda$ that correspond to different choices of e_Λ are equivalent [22]. However, specification of e_Λ is important since the convergence requirement (14) is formulated in terms of norms $||\cdot|| \equiv \sqrt{(\cdot,\cdot)_{e_\Lambda}}$.

(3) It seems to be physically reasonable to choose the vector e_Λ as an eigenvector of the operator $Z_\Lambda^{-1} M_\Lambda^2$ entering the Hamiltonian. Since $Z_\Lambda^{-1} M_\Lambda^2$ is a Hermitian operator with respect to the inner product (12), it has according to the Pontriagin theorem [21] an eigenvector e_Λ such that $\langle e_\Lambda, e_\Lambda\rangle < 0$. Let us find its explicit form. Equation $Z_\Lambda^{-1} M_\Lambda^2 e_\Lambda = \varepsilon_\Lambda e_\Lambda$ is rewritten as

$$m_\Lambda^2 c_\Lambda + g\mu_k^\Lambda \phi_{\Lambda,k} = \varepsilon_\Lambda z_\Lambda c_\Lambda$$
$$g\mu_k^\Lambda c_\Lambda + \Omega_k^2 \phi_{\Lambda,k} = \varepsilon_\Lambda \phi_{\Lambda,k}$$

where $e_\Lambda = (c_\Lambda, \phi_\Lambda)$. Therefore, for $\phi_{\Lambda,k}$ one has

$$\phi_k = \frac{g\mu_k c}{\varepsilon - \Omega_k^2}. \tag{16}$$

The parameter ε_Λ obeys the following equation:

$$\varepsilon_\Lambda z_\Lambda - m_\Lambda^2 = g^2 \sum_k \frac{(\mu_k^\Lambda)^2}{\varepsilon_\Lambda - \Omega_k^2}. \tag{17}$$

For vector (16) $\langle e_\Lambda, e_\Lambda\rangle < 0$ if and only if

$$-b_\Lambda \equiv z_\Lambda + g^2 \sum_k \frac{(\mu_k^\Lambda)^2}{(\varepsilon_\Lambda - \Omega_k^2)^2} < 0. \tag{18}$$

It follows from the Pontriagin theorem [21] that equation (17) has a (real or complex) solution obeying property (18).

Denote by

$$z_{\Lambda,R} = z_\Lambda + \sum_k \frac{(\mu_k^\Lambda)^2}{\Omega_k^4} \qquad m_{\Lambda,R}^2 = m_\Lambda^2 - \sum_k \frac{(\mu_k^\Lambda)^2}{\Omega_k^2} \tag{19}$$

the renormalized values of parameters of the theory.

Equation (17) can be presented in the following form:

$$\sum_k \frac{(\mu_k^\Lambda)^2}{\Omega_k^2}\left(\frac{1}{\varepsilon_\Lambda-\Omega_k^2}+\frac{1}{\Omega_k^2}\right)=z_{\Lambda,R}-m^2_{\Lambda,R}/\varepsilon_\Lambda. \tag{20}$$

It is possible to perform a limit $\Lambda\to\infty$, provided that z_Λ and m^2_Λ are chosen to make $z_{\Lambda,R}$ and $m^2_{\Lambda,R}$ finite as $\Lambda\to\infty$:

$$z_{\Lambda,R}\to z_R \qquad m^2_{\Lambda,R}\to m^2_R.$$

If $\sum_k \mu_k^2/\Omega_k^4=\infty$ but $\sum_k \mu_k^2/\Omega_k^6<\infty$, the infinite renormalization of the wavefunction is indeed necessary, while z_Λ is negative at sufficiently large Λ.

The following cases should be considered.

(i) $m_R^2>0$. Equation (20) has a negative solution $\varepsilon<0$ obeying condition (18). Hamiltonian system (5) is unstable.

(ii) $m_R^2<0$, $z_R>0$. There is an alternative. There may be no real solutions of equation (20) obeying condition (18). There may also be two real negative solutions. The smaller one obeys requirement (18). The Hamiltonian system (5) is also unstable.

(iii) $m_R^2\leqslant 0$, $z_R<0$. Equation (20) may have no real solutions obeying condition (18) or may have a real positive solution satisfying requirement (18). The latter case takes place at sufficiently small $|m_R|^2$. The Hamiltonian system is stable.

Let us consider the most interesting latter case only. Note that the condition $z_R<0$ arises in investigations of large-N QED [41].

If $m_R^2=0$, the formalism of the previous subsection should be slightly modified (the operator $(Z_\Lambda^{-1}M_\Lambda^2)^{-1/4}$ does not exist). For simplicity, consider the case $m_R^2\neq 0$ only.

Let us introduce more convenient coordinates on the Pontriagin space $\mathcal{P}_\Lambda$ in order to remove divergences from Hilbert and indefinite inner products. First of all, present any vector $\psi\in\mathcal{P}_\Lambda$ as

$$\psi=\begin{pmatrix}0\\ \varphi\end{pmatrix}+ce_\Lambda$$

where

$$e_\Lambda=\begin{pmatrix}1\\ \frac{g\mu_k^\Lambda}{\varepsilon^\Lambda-\Omega_k^2}\end{pmatrix}.$$

One finds

$$(e_\Lambda,\psi)=-b_\Lambda c+\sum_{k=1}^\infty \frac{g\mu_k^\Lambda\varphi_k}{\varepsilon_\Lambda-\Omega_k^2}$$

where b has a limit as $\Lambda\to\infty$. Introduce instead of c the new variable $\alpha=-b_\Lambda^{-1}(e_\Lambda,\psi)$:

$$\alpha=c-b_\Lambda^{-1}\sum_{k=1}^\infty \frac{g\mu_k^\Lambda\varphi_k}{\varepsilon_\Lambda-\Omega_k^2}.$$

In terms of new variables $(\alpha;\varphi)$ the inner products (12) and (15) take the form

$$\begin{aligned}\langle\psi,\psi\rangle_\Lambda&=-b|\alpha|^2+\langle\langle\varphi,\varphi\rangle\rangle_\Lambda\\ (\psi,\psi)_{e_\Lambda}&=b|\alpha|^2+\langle\langle\varphi,\varphi\rangle\rangle_\Lambda\end{aligned} \tag{21}$$

with

$$\langle\langle\varphi,\varphi\rangle\rangle_\Lambda=(\varphi,\varphi)+\frac{1}{b_\Lambda}\left|\left(\frac{g\mu}{\varepsilon-\Omega^2},\varphi\right)\right|^2. \tag{22}$$

Formula (22) contains no divergences.

By $\tilde{\mathcal{P}}_\Lambda$ we denote the Pontriagin space of sets (α, φ) with inner products (21). The introduced isomorphism $I_\Lambda : \tilde{\mathcal{P}}_\Lambda \to \mathcal{P}_\Lambda$ has the following form: $I_\Lambda : (\alpha, \varphi) \mapsto (c, \phi)$, where

$$\begin{aligned} c &= \alpha + \frac{1}{b_\Lambda} \sum_{k=1}^{\infty} \frac{g\mu_k^\Lambda \varphi_k}{\varepsilon_\Lambda - \Omega_k^2} \\ \phi_k &= c \frac{g\mu_k^\Lambda}{\varepsilon_\Lambda - \Omega_k^2} + \varphi_k. \end{aligned} \tag{23}$$

(4) It seems to be reasonable to specify the renormalized states by sets $\psi = (\alpha, \varphi)$. By $\tilde{\mathcal{P}}$ we denote the indefinite inner product space of such sets with inner products

$$\begin{aligned} \langle \psi, \psi \rangle &= -b|\alpha|^2 + (\varphi, \varphi) + \frac{1}{b} \left| \left(\frac{g\mu}{\varepsilon - \Omega^2}, \varphi \right) \right|^2 \\ (\psi, \psi) &= b|\alpha|^2 + (\varphi, \varphi) + \frac{1}{b} \left| \left(\frac{g\mu}{\varepsilon - \Omega^2}, \varphi \right) \right|^2 \end{aligned}$$

with $\varepsilon = \lim_{\Lambda \to \infty} \varepsilon_\Lambda$, $b = \lim_{\Lambda \to \infty} b_\Lambda$. However, $\tilde{\mathcal{P}}$ cannot be viewed as a state space. First, the sequence $\frac{g\mu_k}{\varepsilon - \Omega_k^2}$ does not belong to l^2, so one should impose the conditions on φ_k at $k \to \infty$. For example, one can require $\Omega\varphi \in l^2$. Next, the Euclidean space with the inner product $(\cdot, \cdot)$ is not complete, so it is necessary to consider the completeness $\mathcal{P}$ of the space $\tilde{\mathcal{P}}$.

Investigate the explicit form of the space $\mathcal{P}$ (cf [42]).

Let $\{(\alpha^{(n)}, \varphi^{(n)})\}$, $\{(\alpha^{(n)'}, \varphi^{(n)'})\}$ be fundamental sequences. They are equivalent [43] if $||\psi^{(n)} - \psi^{(n)'}|| \to_{n\to\infty} 0$. This means that

$$\begin{aligned} &\alpha^{(n)} - \alpha^{(n)'} \to_{n\to\infty} 0 \\ &||\varphi^{(n)} - \varphi^{(n)'}|| \to_{n\to\infty} 0 \\ &\left(\frac{g\mu}{\Omega^2}; \varphi^{(n)} - \varphi^{(n)'} \right) \to_{n\to\infty} 0. \end{aligned} \tag{24}$$

Furthermore, since the sequence $\{(\alpha^{(n)}, \varphi^{(n)})\}$ is fundamental, sequences $\alpha^{(n)}$, $\varphi^{(n)}$ and $(\frac{g\mu}{\Omega^2}; \varphi^{(n)})$ are also fundamental. Therefore,

$$\begin{aligned} &\alpha^{(n)} \to_{n\to\infty} \alpha \\ &\varphi^{(n)} \to_{n\to\infty} \varphi \\ &\left(\frac{g\mu}{\Omega^2}; \varphi^{(n)} \right) \to_{n\to\infty} \xi. \end{aligned} \tag{25}$$

Thus, two fundamental sequences are equivalent if and only if $\alpha' = \alpha$, $\varphi' = \varphi$ and $\xi' = \xi$.

Let us show now that for any set (α, ξ, φ) there exists a fundamental sequence obeying conditions (25). Note that any sequence obeying requirements (25) is fundamental. It is sufficient to consider two partial cases:

(i) $\varphi = 0$;
(ii) $\alpha = \xi = 0$.

Denote $\chi_k = g\mu_k/(\varepsilon - \Omega_k^2)$. For case (i), set

$$\begin{aligned} &\alpha^{(n)} = \alpha \\ &\varphi_k^{(n)} = \xi \chi_k \Big/ \sum_{k=0}^{n} |\chi_k|^2 \qquad k \leqslant n. \\ &\varphi_k^{(n)} = 0 \qquad k > n. \end{aligned}$$

For case (ii), it is sufficient to check that the set of all vectors $\varphi \in l^2$ satisfying the relations

$$(\chi, \varphi) = 0 \tag{26}$$

is dense in l^2. To prove this property, it is sufficient to notice that any finite vector φ can be approximated by a sequence $\varphi^{(n)} \to \varphi$ obeying requirement (26):

$$\varphi_k^{(n)} = \begin{cases} \varphi_k - \dfrac{(\chi, \varphi)}{\sum_{k=0}^{n} |\chi_k|^n} \chi_k & k \leqslant n \\ \varphi_k & k > n. \end{cases}$$

Thus, the renormalized state space $\mathcal{P}$ is a space of sets (α, φ, ξ), where $\varphi \in l^2$, $\alpha \in \boldsymbol{C}$, $\xi \in \boldsymbol{C}$.

The following inner products are introduced in $\mathcal{P}$:

$$\begin{aligned} (\psi, \psi) &= b|\alpha|^2 + (\varphi, \varphi) + b^{-1}|\xi|^2 \\ \langle \psi, \psi \rangle &= -b|\alpha|^2 + (\varphi, \varphi) + b^{-1}|\xi|^2. \end{aligned} \tag{27}$$

(5) Let us construct the mapping $\tilde{P}_\Lambda : \mathcal{P} \to \tilde{\mathcal{P}}_\Lambda$ which transforms the renormalized state (α, ξ, φ) to the regularized state $(\alpha_\Lambda, \varphi_\Lambda)$. Choose it in such a way that

$$\begin{aligned} &\alpha_\Lambda \to_{\Lambda\to\infty} \alpha \qquad ||\varphi_\Lambda - \varphi|| \to_{\Lambda\to\infty} 0 \\ &\left(\frac{g\mu^\Lambda}{\varepsilon^\Lambda - \Omega^2}, \varphi_\Lambda \right) \to_{\Lambda\to\infty} \xi. \end{aligned} \tag{28}$$

The mapping $P_\Lambda : \mathcal{P} \to \mathcal{P}_\Lambda$ will have the form $P_\Lambda = I_\Lambda \tilde{P}_\Lambda$.

The following proposition is a direct corollary of equations (21).

Proposition 1. *Let* $\tilde{P}_\Lambda : (\alpha, \xi, \varphi) \mapsto (\alpha_\Lambda, \varphi_\Lambda)$ *be a mapping satisfying requirements (28). Then*

$$||(\tilde{\alpha}_\Lambda, \tilde{\varphi}_\Lambda) - \tilde{P}_\Lambda(\alpha, \xi, \varphi)|| \to_{\Lambda\to\infty} 0$$

if and only if $(\tilde{\alpha}_\Lambda, \tilde{\varphi}_\Lambda)$ *obeys requirements (28).*

Proposition 1 tells us that the form of operator T_Λ obeying requirements (28) is not important.

By $Q_\Lambda : \tilde{\mathcal{P}}_\Lambda \to \mathcal{P}$ we denote the operator $Q_\Lambda : (\alpha_\Lambda, \varphi_\Lambda) \mapsto (\alpha'_\Lambda, \xi'_\Lambda, \varphi'_\Lambda)$ of the form

$$\alpha'_\Lambda = \alpha_\Lambda \qquad \varphi'_\Lambda = \varphi_\Lambda \qquad \xi'_\Lambda = \left(\frac{g\mu^\Lambda}{\varepsilon^\Lambda - \Omega^2}, \varphi_\Lambda \right).$$

Proposition 1 can be reformulated as follows:

Proposition $1'$**.** *Let* $(\alpha_\Lambda, \varphi_\Lambda) \in \tilde{\mathcal{P}}_\Lambda$. *Then* $||(\alpha_\Lambda, \varphi_\Lambda)|| \to 0$ *if and only if* $Q_\Lambda(\alpha_\Lambda, \varphi_\Lambda) \to_{\Lambda\to\infty} 0$.

Equation (28) also implies:

Proposition 2.

$$s - \lim_{\Lambda\to\infty} Q_\Lambda \tilde{P}_\Lambda = 1. \tag{29}$$

We will also require that

$$\langle \tilde{P}_\Lambda \psi, \tilde{P}_\Lambda \tilde{\psi} \rangle \to_{\Lambda\to\infty} \langle \psi, \tilde{\psi} \rangle \qquad ||\tilde{P}_\Lambda|| \leqslant A = \text{const} \tag{30}$$

for some Λ-independent a.

The explicit form of the mapping $\tilde{P}_\Lambda : (\alpha, \xi, \varphi) \mapsto (\alpha_\Lambda, \varphi_\Lambda)$ can be chosen as

$$\varphi_\Lambda = \varphi + \frac{\chi^\Lambda(\xi - (\chi^\Lambda, \varphi))}{(\chi^\Lambda, \chi^\Lambda)} \qquad \alpha_\Lambda = \alpha$$

where $\chi^\Lambda = \frac{g\mu^\Lambda}{\varepsilon^\Lambda - \Omega^2}$. The properties $\alpha_\Lambda \to \alpha$, $(\chi^\Lambda, \varphi^\Lambda) = \xi \to \xi$ at $\Lambda \to \infty$ are evident. Since $||\chi^\Lambda|| \to \infty$, one has

$$s - \lim_{\Lambda\to\infty} \frac{\chi^\Lambda}{(\chi^\Lambda, \chi^\Lambda)} = 0.$$

It is sufficient then to check that

$$\lim_{\Lambda\to\infty} \frac{(\chi^\Lambda, \varphi)}{||\chi^\Lambda||} \qquad \varphi \in l^2. \tag{31}$$

For finite sequences φ, property (31) is evident. It follows from the standard theorems of functional analysis [44] that property (31) is then satisfied for all $\varphi \in l^2$.

(6) To check property (14), it is convenient to investigate the resolvent of the operator $Z_\Lambda^{-1} M_\Lambda^2$. To find its explicit form,

$$(\lambda + Z_\Lambda^{-1} M_\Lambda^2)^{-1} : \begin{pmatrix} c \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} c \\ \phi \end{pmatrix}$$

one should solve the system of equations

$$(\lambda z_\Lambda + m_\Lambda^2)c' + g\mu^\Lambda \phi' = z_\Lambda c$$
$$g\mu^\Lambda c' + (\lambda + \Omega^2)\phi' = \phi.$$

Therefore,

$$c' = a_\Lambda \left[z_\Lambda c - g \sum_{k=1}^{\infty} \frac{\mu_k^\Lambda \phi_k}{\lambda + \Omega_k^2} \right]$$
$$\phi' = \frac{1}{\lambda + \Omega^2}\phi - \frac{g\mu^\Lambda}{\lambda + \Omega^2} c'$$

with

$$a_\lambda = \left(\lambda z_\Lambda + m_\Lambda^2 - \sum_k \frac{g^2 (\mu_k^\Lambda)^2}{\lambda + \Omega_k^2} \right)^{-1}.$$

Making use of the (α, φ)-coordinates, one obtains

$$\alpha' = \frac{1}{\lambda + \varepsilon_\Lambda} \alpha$$
$$\varphi' = \frac{1}{\lambda + \Omega^2}\varphi + \frac{g\mu^\Lambda (\varepsilon_\Lambda + \lambda) a_\Lambda}{(\varepsilon_\Lambda - \Omega^2)(\lambda + \Omega^2)} \left(\frac{g\mu^\Lambda}{\lambda + \Omega^2}; \varphi \right)$$

(equation (17) is taken into account). Thus, the explicit form of the operator $I_\Lambda^{-1}(\lambda + Z_\Lambda^{-1} M_\Lambda^2)^{-1} I_\Lambda : (\alpha, \varphi) \mapsto (\alpha', \varphi')$ is found. This operator is Hermitian at real values of λ with respect to Hilbert and indefinite inner products (21).

(7) Investigate now the behaviour of the resolvent at $\Lambda \to \infty$. Consider the operator $Q_\Lambda I_\Lambda^{-1}(\lambda + Z_\Lambda^{-1} M_\Lambda^2)^{-1} I_\Lambda : (\alpha, \varphi) \mapsto (\alpha', \xi', \varphi')$, which can be presented as

$$\alpha' = \alpha$$
$$\varphi' = \frac{1}{\lambda + \Omega^2}\varphi - \frac{g\mu^\Lambda (\varepsilon_\Lambda + \lambda) a_\Lambda}{(\varepsilon_\Lambda - \Omega^2)(\lambda + \Omega^2)}\xi + \frac{g\mu^\Lambda (\varepsilon_\Lambda + \lambda)^2 a_\Lambda}{(\varepsilon_\Lambda - \Omega^2)(\lambda + \Omega^2)} \left(\frac{g\mu^\Lambda}{(\varepsilon_\Lambda - \Omega^2)(\lambda + \Omega^2)}, \varphi \right) \tag{32}$$
$$\xi' = -(\lambda + \varepsilon_\Lambda) a_\Lambda b_\Lambda \left(\frac{g\mu^\Lambda}{(\varepsilon_\Lambda - \Omega^2)(\lambda + \Omega^2)}, \varphi \right) - \sum_{k=1}^{\infty} \frac{g^2 (\mu_k^\Lambda)^2 (\varepsilon_\Lambda + \lambda) a_\Lambda}{(\varepsilon_\Lambda - \Omega^2)^2 (\lambda + \Omega^2)} \xi$$

(equation (17) is used), where $\xi = (\frac{g\mu^\Lambda}{\varepsilon_\Lambda - \Omega^2}, \varphi)$.

Denote the mapping (32) as $(\lambda + H_\Lambda)^{-1} : (\alpha, \xi, \varphi) \mapsto (\alpha', \xi', \varphi')$. It is a resolvent of a positive self-adjoint operator. We have obtained the following proposition.

Proposition 3. *The following relation is satisfied:*

$$Q_\Lambda I_\Lambda^{-1}(\lambda + Z_\Lambda^{-1} M_\Lambda^2)^{-1} I_\Lambda = (\lambda + H_\Lambda)^{-1} Q_\Lambda.$$

We see that the operator $(\lambda + H_\Lambda)^{-1}$ has a strong limit $(\lambda + H)^{-1}$, being a resolvent of a positive self-adjoint operator. General results of [39,45] tell us that the following statement is satisfied.

Proposition 4. *Let f be a bounded Borel function. Then*

$$s - \lim_{\Lambda\to\infty} f(H_\Lambda) = f(H).$$

Proposition 3 also implies that:

Proposition 5. *For any bounded Borel function $f : \boldsymbol{R} \to \boldsymbol{R}$*

$$Q_\Lambda I_\Lambda^{-1} f(Z_\Lambda^{-1} M_\Lambda^2) I_\Lambda = f(H_\Lambda) Q_\Lambda.$$

Let us prove relation (14).

Proposition 6. *For any bounded Borel function $f : \boldsymbol{R} \to \boldsymbol{R}$ and any $\psi \in \mathcal{P}$*

$$||f(Z_\Lambda^{-1} M_\Lambda^2) P_\Lambda \psi - P_\Lambda f(H)\psi|| \to_{\Lambda\to\infty} 0.$$

Proof. Since $I_\Lambda : \tilde{\mathcal{P}}_\Lambda \to \mathcal{P}_\Lambda$ is an isomorphism, proposition $1'$ implies that relation (3) is satisfied if and only if

$$||Q_\Lambda I_\Lambda^{-1} f(Z_\Lambda^{-1} M_\Lambda^2) I_\Lambda \tilde{P}_\Lambda \psi - Q_\Lambda \tilde{P}_\Lambda f(H)\psi|| \to_{\Lambda\to\infty} 0.$$

It follows from proposition 5 that this relation can be rewritten as

$$||f(H_\Lambda) Q_\Lambda \tilde{P}_\Lambda \psi - Q_\Lambda \tilde{P}_\Lambda f(H)\psi|| \to_{\Lambda\to\infty} 0. \tag{33}$$

It follows from propositions 2 and 4 that $s - \lim_{\Lambda\to\infty} f(H_\Lambda) Q_\Lambda \tilde{P}_\Lambda = f(H)$, i.e.

$$||f(H_\Lambda) Q_\Lambda \tilde{P}_\Lambda \psi - f(H)\psi|| \to_{\Lambda\to\infty} 0. \tag{34}$$

Proposition 2 also implies that

$$||f(H)\psi - Q_\Lambda \tilde{P}_\Lambda f(H)\psi|| \to_{\Lambda\to\infty} 0. \tag{35}$$

Combining equations (34) and (35), we obtain relation (33). Proposition 6 is proved. □

(8) Thus, we have constructed the renormalized state space $\mathcal{L} = \mathcal{F}(\mathcal{P})$. The 'one-particle' renormalized space is chosen to be a space of sets (α, ξ, φ) with the inner products (27); the Hamiltonian operator is also defined by specifying the resolvent $(\lambda + H)^{-1}$. The evolution operator $U(t)$ entering equation (14) has the form

$$U(t) = \oplus_{n=0}^{\infty} (\mathrm{e}^{-\mathrm{i}H^{1/2}t})^{\otimes n}.$$

4. 'Field' operators

Let us construct now Heisenberg field operators $Q(t)$, $q_k(t)$ and their linear combinations in the renormalized theory. According to equations (6), they should be expressed via creation and annihilation operators. Let us recall their definition (see, e.g., [1]).

The set of all vectors $\psi^{\otimes n}$ is a total set in $\mathcal{P}^{\vee n}$. By $b_n^-(\gamma) : \mathcal{P}^{\vee n} \to \mathcal{P}^{\vee(n-1)}$, $b_n^+(\gamma) : \mathcal{P}^{\vee(n-1)} \to \mathcal{P}^{\vee n}$, $\gamma \in \mathcal{P}$, we denote the linear operators which are uniquely defined from the relations

$$b_n^-(\gamma)\psi^{\otimes n} = \sqrt{n}(\gamma, \psi)\psi^{\otimes(n-1)}$$

$$b_n^+(\gamma)\psi^{\otimes(n-1)} = n^{-1/2} \sum_{j=0}^{n-1} \psi^{\otimes j} \otimes \gamma \otimes \psi^{\otimes(n-1-j)}.$$

Moreover, $||b_n^{\pm}(\gamma)|| \leqslant n^{1/2}||\gamma||$. By $b^{\pm}(\gamma) : \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P})$ we denote the operators

$$(b^+(\gamma)\Psi)_n = b_n^+(\gamma)\Psi_{n-1} \qquad (b^-(\gamma)\Psi)_{n-1} = b_n^+(\gamma)\Psi_n$$

which are defined on the set of all finite vectors of the Fock space.

Proposition 7. *The following relations are satisfied:*

$$b^+(P_\Lambda\gamma)T_\Lambda = T_\Lambda b^+(\gamma) \qquad b^-(\gamma_\Lambda)T_\Lambda = T_\Lambda b^-(P_\Lambda^+\gamma_\Lambda)$$

for arbitrary $\gamma_\Lambda \in \mathcal{P}_\Lambda$, $\gamma \in \mathcal{P}$.

The proof is straightforward.

Proposition 8. *Let* $\gamma_\Lambda \in \mathcal{P}_\Lambda$, $\gamma \in \mathcal{P}$, Ψ *is a finite vector of the renormalized Fock space* $\mathcal{F}(\mathcal{P})$ *and*

$$||\gamma_\Lambda - P_\Lambda\gamma|| \to_{\Lambda\to\infty} 0. \tag{36}$$

Then

$$||b^{\pm}(\gamma_\Lambda)T_\Lambda - T_\Lambda b^{\pm}(\gamma))\Psi|| \to_{\Lambda\to\infty} 0. \tag{37}$$

Proof. For the creation operator, equation (37) means that

$$||b^+(\gamma_\Lambda - P_\Lambda\gamma)\Psi|| \to_{\Lambda\to\infty} 0.$$

Let $\Psi = (\psi_0, \psi_1, \ldots, \psi_n, 0, 0, \ldots)$. Then

$$||b^+(\gamma_\Lambda - P_\Lambda\gamma)\Psi|| \leqslant \max(1, ||P_\Lambda||^n)||\Psi||n^{1/2}||\gamma_\Lambda - P_\Lambda\gamma|| \to_{\Lambda\to\infty} 0.$$

For the annihilation operator, it is necessary to check that

$$||T_\Lambda b^-(P_\Lambda^+\gamma_\Lambda - \gamma)\Psi|| \to_{\Lambda\to\infty} 0 \tag{38}$$

for $\Psi \in \mathcal{P}^{\vee n}$. The Banach–Steinhaus theorem [44] implies that it is sufficient to prove property (38) for $\Psi = \psi^{\otimes n}$, $\psi \in \mathcal{P}$. This property is correct if and only if

$$(P_\Lambda^+\gamma_\Lambda - \gamma, \psi) \to_{\Lambda\to\infty} 0$$

i.e. $(\gamma_\Lambda, P_\Lambda\psi) \to_{\Lambda\to\infty} (\gamma, \psi)$. Equations (36) and (30) confirm this property. The proposition is proved. □

Propositions 1′ and 8 imply the following corollary.

Proposition 9. *Let* $\gamma_\Lambda \in \mathcal{P}_\Lambda$, $\gamma \in \mathcal{P}$, Ψ *is a finite vector of the renormalized Fock space* $\mathcal{F}(\mathcal{P})$ *and*

$$||Q_\Lambda I_\Lambda^{-1}\gamma_\Lambda - \gamma|| \to_{\Lambda\to\infty} 0. \tag{39}$$

Then property (37) is satisfied.

We see that the operator $b^{\pm}(\gamma_\Lambda)$ in the regularized theory corresponds to the operator $b^{\pm}(\gamma)$ in the renormalized theory. One can say $b^{\pm}(\gamma_\Lambda) \to b^{\pm}(\gamma)$ in a generalized strong sense [39, 40].

Note that the linear combinations $\sum_k b_k^+\zeta_k$ and $\sum_k b_k^-\zeta_k^*$ of the operators $b_k^{\pm}$ (6) can be presented as $b^+(\zeta)$ and $b^-(\zeta)$ correspondingly with $\zeta \in \mathcal{P}_\Lambda$.

Consider the linear combination $\sum_{k=0}^{\infty} q_k(t)\chi_k^\Lambda$ in the regularized theory. It follows from equations (6) that

$$\sum_{k=0}^{\infty} q_k(t)\chi_k^\Lambda = b^+(\gamma_\Lambda^t) + b^-(\gamma_\Lambda^t)$$

with

$$\gamma_\Lambda^t = \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}(Z_\Lambda^{-1}M_\Lambda^2)^{1/2}t}(Z_\Lambda^{-1}M_\Lambda^2)^{-1/4}Z_\Lambda^{-1}\chi_\Lambda.$$

Propositions 4, 5 and 9 imply the following statement.

Proposition 10. *Let*

$$Q_\Lambda I_\Lambda^{-1} Z_\Lambda^{-1} \chi_\Lambda \to_{\Lambda\to\infty} \gamma. \tag{40}$$

Ψ *is a finite vector of the renormalized Fock space* $\mathcal{F}(\mathcal{P})$. *Then*

$$\left\| \left(\sum_{k=0}^{\infty} q_k(t) \chi_k^\Lambda T_\Lambda - T_\Lambda q_t(\gamma) \right) \Psi \right\| \to_{\Lambda\to\infty} 0$$

with

$$q_t(\gamma) = b^+ \left(\frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}H^{1/2}t} H^{-1/4} \gamma \right) + b^- \left(\frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}H^{1/2}t} H^{-1/4} \gamma \right).$$

Let us write down the explicit form of condition (40). Let $\chi_\Lambda = \begin{pmatrix} c_\Lambda \\ \phi_\Lambda \end{pmatrix}$, $\gamma = (\alpha, \xi, \varphi)$,

$$\varphi_\Lambda = \phi_\Lambda - z_\Lambda^{-1} c_\Lambda \frac{g\mu^\Lambda}{\varepsilon_\Lambda - \Omega^2}.$$

Equation (40) means that

$$||\varphi_\Lambda - \varphi|| \to_{\Lambda\to 0} 0 \qquad \left(\frac{g\mu^\Lambda}{\varepsilon_\Lambda - \Omega^2} \right) \to_{\Lambda\to 0} \xi \qquad z_\Lambda^{-1} c_\Lambda \to_{\Lambda\to 0} \alpha + \xi/b. \tag{41}$$

We see that the operator $\sum_{k=0}^{\infty} q_k(t)\gamma_k^\Lambda$ in the regularized theory corresponds to the operator $q(\gamma)$ in the renormalized theory, provided that requirements (41) are satisfied.

Example 1. Let $\Omega\phi \in l^2$, $\Omega\phi_\Lambda \to \Omega\phi$, $c_\Lambda = 0$. Then the expression $\sum_{k=1}^{\infty} q_k(t)\phi_k$ corresponds to the operator $q(\alpha, \xi, \phi)$ in the renormalized theory, where

$$\alpha = -\frac{\xi}{b} \qquad \xi = \left(\frac{g\mu}{\Omega(\varepsilon - \Omega^2)}, \Omega\phi \right) \qquad \varphi = \phi.$$

Example 2. Let $c_\Lambda = 1$, $\phi_\Lambda = 0$. Then the expression $\sum_{k=0}^{\infty} q_k(t)\gamma_k^\Lambda$ takes the form $Q(t)$. For this case $\alpha + \xi/\beta = 0$,

$$\varphi_\Lambda = -z_\Lambda \frac{g\mu^\Lambda}{\varepsilon_\Lambda - \Omega^2}$$

so that $||\varphi_\Lambda|| \to_{\Lambda\to\infty} 0$,

$$\left(\frac{g\mu^\Lambda}{\varepsilon_\Lambda - \Omega^2}, \varphi_\Lambda \right) = -z_\Lambda^{-1}(-b_\Lambda - z_\Lambda) \to_{\Lambda\to\infty} 1.$$

Thus, $\xi = 1$, so that $\alpha = -1/\xi$. We see that $Q(t) \to q_t(-1/b, 1, 0)$.

Example 3. Let $c_\Lambda = z_\Lambda$, $\phi_\Lambda = \frac{g\mu^\Lambda}{\varepsilon_\Lambda - \Omega^2}$. Then $\varphi_\Lambda = 0$, $\xi = 0$, $\alpha + \xi/b = 1$. Thus, the expression

$$z_\Lambda Q(t) + \sum_{k=1}^{\infty} \frac{g\mu_k^\Lambda q_k(t)}{\varepsilon_\Lambda - \Omega_k^2}$$

corresponds to the operator $q_t(1, 0, 0)$ in the renormalized theory.

Combining examples 1–3, we find that the expressions (2) correspond to correctly defined operators in the renormalized theory.

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